On solitary waves in classical anisotropic Heisenberg chains with generalized boundary conditions

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Abstract

We examine solitary waves in classical ferromagnetic Heisenberg chains with an uniaxial anisotropy and a parallel magnetic field in a continuum approach. The boundary conditions commonly used are generalized to nonlinear spin wave states, which themselves turn out to be stable only for an anisotropy of the easy–plane type. In this case we obtain two different branches of one–soliton–solutions which can be mapped onto each other by a formal time inversion. Moreover, they show some remarkable similarity to dark solitons of the Nonlinear Schrödinger equation. Numerical simulations for the discrete Heisenberg chain show that these solitary waves are highly, but not absolutely stable under interaction with linear excitations and as well under scattering with each other. The possible significance of these solitary waves in a phenomenological theory of one–dimensional magnets is briefly addressed.
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1 The model

The discrete classical ferromagnetic Heisenberg chain with a local uniaxial anisotropy and a magnetic field parallel to this axis is given by

$$\tilde{\mathcal{H}} = -\sum_n \left[\vec{S}_n\vec{S}_{n+1} + hS_n^z + \frac{\alpha}{2}\left(S_n^z\right)^2\right] . \tag{1}$$

The classical spins are unit vectors $\vec{S}_n = (\sin\vartheta_n\cos\varphi_n, \sin\vartheta_n\sin\varphi_n, \cos\vartheta_n)$, h the magnetic field and α the anisotropy parameter. If we choose the infinite chain to lie in the x–direction and an appropriate length unit with lattice spacing $a = 1$, the continuum approximation of (1) in lowest order reads

$$\mathcal{H} = \int_{\infty}^{\infty} dx \left(\frac{1}{2}\left(\frac{(\partial_x p)^2}{1-p^2} + \left(1-p^2\right)(\partial_x q)^2\right) - hp - \frac{\alpha}{2}p^2\right) . \tag{2}$$

Here the classical spins are described by the canonical conjugate fields $p(x,t) = \cos(\vartheta(x,t))$, $q(x,t) = \varphi(x,t)$. The equations of motion are

$$\begin{aligned}
\partial_t q &= -\frac{1}{1-p^2}\partial_x^2 p - \frac{p}{(1-p^2)^2}(\partial_x p)^2 - p(\partial_x q)^2 - h - \alpha p\,, \qquad (3)\\
\partial_t p &= (1-p^2)\partial_x^2 q - 2p(\partial_x p)\partial_x q\,. \qquad (4)
\end{aligned}$$

For $\alpha < 0$ and $|\frac{h}{\alpha}| < 1$ the solution of lowest energy is given by constant q and $p = -\frac{h}{\alpha}$ (easy–plane or, for finite h, easy–cone model), for $\alpha > 0$ by $|p| = 1$ (easy–axis model). These field configurations can be used as boundary conditions for the above equations. In the literature only these two cases appear to be considered, for a review see [1, 2].
Takhtajan [3] and Fogedby [4] have shown that the isotropic model ($\alpha = 0$) allows for a Lax representation and is solvable by the inverse scattering method. Moreover, this model is integrable in the sense that it has an infinite series of independent conserved quantities. These results have been extended by Sklyanin [5] to the case of a general biaxial anisotropy. The relationship of the model (2) to the Nonlinear Schrödinger equation has also been established for the isotropic [6, 7] and the anisotropic case [8, 9]. However, all these results have been obtained using boundary conditions of the above type. In the present work we examine the model for more general boundary conditions to be specified below.
If the spin configuratiuon is the same at both boundaries, the system has (at least) two well-known conserved quantities apart from the energy, namely the total momentum

$$P = \int dx\, p \partial_x q \tag{5}$$

and the z–component of the total angular momentum or magnetic moment

$$M = \int dx\, p\,. \tag{6}$$

P is the generator of translations and M is the generator of uniform rotations of the spins. The Poisson brackets of $\mathcal{H}$, P, M with each other vanish since surface terms do not contribute under the above condition.

2 Nonlinear spin waves and linear magnons

From the above equations one easily finds the solutions

$$p = u \quad , \quad q = q_0 + kx - \omega t \tag{7}$$

with constants u, k, q_0, $|u| < 1$, and ω given by

$$\omega = \left(k^2 + \alpha\right) u + h\,. \tag{8}$$

Here the spin field describes a propagating wave characterized by the parameters u and k. (7) is an exact solution of the nonlinear equations of motion and will therefore be called a nonlinear spin wave. It has a homogenous energy density obtained from (2) as

$$\varepsilon = \frac{1}{2}\left(1 - u^2\right) k^2 - hu - \frac{\alpha}{2} u^2\,. \tag{9}$$

Note that the spin configurations mentioned in the previous section are included as special cases. The time–dependent field configuration (7) can be used as a generalization of the boundary conditions given above. In the next section we will obtain solitary waves obeying such boundary conditions, i. e. having the asymptotic form of a spin wave for $|x| \to \infty$. As the spin waves in general do not give absolute minima of the Hamiltonian their stability is

to be questioned. Therefore let us briefly discuss the stability of the homogenous spin wave solution (7) if the system is forced to have the asymptotic structure

$$\lim_{|x|\to\infty} p = u \quad , \quad \lim_{|x|\to\infty} \partial_x q = k \,. \tag{10}$$

The ansatz

$$p(x,t) = u + \eta(x,t) \quad , \quad q(x,t) = q_0 + kx - \omega t + \xi(x,t) \tag{11}$$

with η, ξ vanishing for $|x| \to \infty$ describes small fluctuations around the spin wave solution respecting the boundary condition and the conservation of P and M. Introducing the Fourier transforms

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int dl\, \tilde{\eta}(l,t) e^{\imath lx} \,, \tag{12}$$

$$\xi(x,t) = \frac{1}{\sqrt{2\pi}} \int dl\, \tilde{\xi}(l,t) e^{\imath lx} \tag{13}$$

this leads to the restriction $\tilde{\eta}(0,t) = 0$. Inserting these expressions in the equations of motion and linearizing in η, ξ gives

$$\partial_t \tilde{\eta} = -2\imath ukl\tilde{\eta} - l^2 \left(1 - u^2\right) \tilde{\xi} \,, \tag{14}$$

$$\partial_t \tilde{\xi} = \left(\frac{l^2}{1-u^2} - k^2 - \alpha\right) \tilde{\eta} - 2\imath ukl\tilde{\xi} \,. \tag{15}$$

The general solution of these equations has the form $\tilde{\eta}(l,t) = \tilde{\eta}_0(l) e^{-\imath\omega_l t}$, $\tilde{\xi}(l,t) = \tilde{\xi}_0(l) e^{-\imath\omega_l t}$ with

$$\omega_l = 2ukl \pm \sqrt{l^4 + l^2 \left(1-u^2\right)\left(-\alpha - k^2\right)} \,. \tag{16}$$

The nonlinear spin wave solution is stable if ω_l is real for every Fourier mode l. This is the case either for $|u| = 1$, or for

$$-\alpha - k^2 \geq 0 \tag{17}$$

with general u. The relation (17) can only be fullfilled for finite k if $\alpha < 0$, and stable exact spin wave solutions (7) are restricted to such type of Hamiltonian. If (17) is valid eq. (16) is the spectrum of linear excitations above the nonlinear spin wave. We shall call these excitations magnons. The extremal group velocities within the two magnon dispersion branches are achieved at $l = 0$ with

$$\left(\frac{d\omega_l}{dl}\right)_{l=0} = 2uk \pm \sqrt{\left(1-u^2\right)\left(-\alpha - k^2\right)} \,. \tag{18}$$

The upper sign corresponds to a minimum, the lower to a maximum of $d\omega_l/dl$ on the particular branch.

3 Solitary waves

As the nonlinear spin waves are stable uniform solutions under the condition (17) we use them as a reference state to construct localized solutions. We are looking for solitary solutions which

have the form of a spin wave for $|x| \to \infty$ determined by parameters u, k.
Inserting the ansatz

$$p(x,t) = p(x - vt) \quad , \quad q(x,t) = \Omega t + \bar{q}(x - vt) \tag{19}$$

and performing two formal integrations one ends up with (cf. Fogedby [4])

$$\frac{d\bar{q}}{dx} = v\frac{r-p}{1-p^2}\,, \tag{20}$$

$$\left(\frac{dp}{dx}\right)^2 = -v^2\left(1 + r^2 - 2rp\right) + 2\left(h + \Omega\right)p\left(p^2 - 1\right) - s\left(p^2 - 1\right) + \alpha p^2\left(p^2 - 1\right)\,, \tag{21}$$

where r, s are integration constants, v the velocity of the solitary wave and Ω an internal frequency. Denoting the r.h.s. of (21) by $F(p)$ we conclude that this quantity should be nonnegative for certain values of p between (-1) and 1. A real single root of $F(p)$ corresponds to an extremal value of $p(x)$, while a real double root $p = u$ leads to a fixed point since one can easily show that in this case also the second and successively all higher derivatives of p vanish for $p = u$. For our solution to be localized and asymptotically in the state $p = u$ we need the latter case and make the general ansatz

$$\left(\frac{dp}{dx}\right)^2 = \alpha(p-u)^4 + \beta(p-u)^3 + \gamma(p-u)^2\,. \tag{22}$$

To fullfill the additional boundary condition for q we must also have

$$\lim_{|x|\to\infty} \partial_x \bar{q} = v\frac{r-u}{1-u^2} = k\,. \tag{23}$$

A comparision of eqs. (20), (21) and (22), (23) leads to

$$vr = k\left(1 - u^2\right) + uv\,, \tag{24}$$

$$s = v^2 + 2kuv + k^2 - 3u^2k^2 - \alpha u^2 \tag{25}$$

and

$$\beta = -2u\left(-\alpha - k^2\right) + 2k\left(v - 2uk\right)\,, \tag{26}$$

$$\gamma = -\left(v - 2uk\right)^2 + \left(-\alpha - k^2\right)\left(1 - u^2\right)\,, \tag{27}$$

where the parameters v and Ω are related by

$$vk = \left(k^2 + \alpha\right)u + h + \Omega\,. \tag{28}$$

The additional roots $u_\pm$ of $F(p)$ are given by

$$\begin{aligned} u_\pm - u &= -\frac{\beta}{2\alpha} \pm \sqrt{\frac{\beta^2}{4\alpha^2} - \frac{\gamma}{\alpha}} \\ &= -u + \frac{uk^2}{\alpha} - \frac{vk}{\alpha} \pm \sqrt{\frac{-\alpha - (v - uk)^2}{\alpha^2}\left(-\alpha - k^2\right)} \end{aligned} \tag{29}$$

3.1 The case $\alpha < 0$

The possible behavior of $F(p)$ for $\alpha < 0$ and real $u_\pm$ is shown in figure 1. For a localized solution we need $u_+ \geq u \geq u_-$ (a), while otherwise, e. g. case (b), one obtains a periodic wave train oscillating between u_+ and u_-. If $\alpha < 0$ a necessary and sufficient condition for $u_+ \geq u \geq u_-$ is $\gamma \geq 0$, which also ensures the reality of $u_\pm$. Thus we have

$$\gamma = -\left(v - 2uk\right)^2 + \left(-\alpha - k^2\right)\left(1 - u^2\right) \geq 0 \tag{30}$$

as a restriction to the soliton parameter v. Moreover, the desired solution is only meaningful if it holds $|p(x)| \leq 1$ for all x, i. e. $|u_\pm| \leq 1$. Using $\alpha < 0$ one can derive from the above equations that $u_+ \leq 1$ is equivalent to $(v - k(1+u))^2 \geq 0$ and $u_- \geq -1$ is equivalent to $(v - k(1-u))^2 \geq 0$. Thus, for every choice of the soliton parameter v compatible with (30) we have two solitary solutions which can be obtained explicitly by elementary integration of eqs. (22), (20):

$$p_\pm(x,t) = u + \frac{\gamma\left(u_\pm - u\right)}{\left(\gamma + \frac{\beta}{2}\left(u_\pm - u\right)\right)\cosh\left(\sqrt{\gamma}\left(x - vt - x_0\right)\right) - \frac{\beta}{2}\left(u_\pm - u\right)}\,, \tag{31}$$

$$\begin{aligned} q_\pm(x,t) &= q_0 + kx - \omega t \\ &\quad - \tan^{-1}\left(\frac{v + k\left(1-u\right)}{\sqrt{\gamma}}\frac{u_\pm - u}{1 + u_\pm}\tanh\left(\frac{\sqrt{\gamma}}{2}\left(x - vt - x_0\right)\right)\right) \\ &\quad - \tan^{-1}\left(\frac{v - k\left(1+u\right)}{\sqrt{\gamma}}\frac{u_\pm - u}{1 - u_\pm}\tanh\left(\frac{\sqrt{\gamma}}{2}\left(x - vt - x_0\right)\right)\right) \end{aligned} \tag{32}$$

The double sign corresponds to a different choice of the integration constant $p(x_0, 0) = u_\pm$, and q_0, x_0 are further constants. The two solutions are related via a formal time inversion, i. e. the mapping $(x,t) \mapsto (x,-t)$, $(q,p) \mapsto (q,-p)$, $(h,\alpha) \mapsto (-h,\alpha)$, $(k,u) \mapsto (k,-u)$, $(v,\Omega) \mapsto (-v,-\Omega)$. As a consequence we have $(\beta,\gamma) \mapsto (-\beta,\gamma)$, $u_\pm \mapsto -u_\mp$, and the $(+)$– and the $(-)$– solution interchange. Note also that the Hamiltonian and the equations of motion keep their form under such operation.

We have found the general solitary wave of the form (19) obeying the boundary conditions (10). The solutions for p are pulse solitons with amplitude $(u_\pm - u)$ and width $1/\sqrt{\gamma}$ which are parametrized by the velocity v; a typical example is given in figure 2. From (31), (32) one explitly sees the structure of a spin wave (7) for $|x| \gg 1/\sqrt{\gamma}$. In this sense these solutions may be called dark solitons. The phase velocity of the spin wave and the soliton velocity differ by Ω/k (cf. (8), (28)), which can alternatively be used as soliton parameter. Note also that the range of admissible velocities given by (30) is centered around the formal group velocity of the spin wave. Moreover, as seen from eq. (18) the maximum (minimum) soliton velocity is given by the minimum (maximum) group velocity of the linear magnons on the upper (lower) magnon branch.

The above results are in agreement with earlier work by other authors, beginning with Akhiezer and Borovik [10], who took the absolute ground state of the model as boundary condition.

For $|u| < 1$ the inequality (30) can only be fullfilled if

$$-\alpha - k^2 \geq 0\,. \tag{33}$$

This is identical with the condition (17) for the stability of the spin waves. Thus, the above pair of one–soliton–solutions exists exactly in the region of boundary parameters u, k where the asymptotic structure of the solutions is found to be stable.
If v takes its extremal values, i. e. $\gamma = 0$, and assuming $\beta > 0$, the $(-)$–soliton becomes a simple spin wave, while the $(+)$–soliton gets an algebraic structure:

$$\begin{aligned} p_+(x,t) &= u + \frac{\frac{\beta}{-\alpha}}{1 + \frac{\beta^2}{-4\alpha}(x - vt - x_0)^2} , && (34) \\ q_+(x,t) &= q_0 + kx - \omega t \\ &\quad - \tan^{-1}\left(\frac{\beta\,(v + k\,(1-u))}{-\alpha + \beta - \alpha u}\frac{(x - vt - x_0)}{2}\right) \\ &\quad - \tan^{-1}\left(\frac{\beta\,(v - k\,(1+u))}{-\alpha - \beta + \alpha u}\frac{(x - vt - x_0)}{2}\right) && (35) \end{aligned}$$

If $\beta < 0$ for an extremal value of v the above results hold vice versa. An algebraic form of a soliton in such limiting cases has also been obtained by Ivanov *et al.* [11] considering the special case $u = \frac{h}{-\alpha}$, $k = 0$.
The energy density of the solitary waves is given by

$$\varepsilon_\pm = -\alpha\,(p_\pm - u)^2 + (-k\,(v - uk) - h - \alpha u)\,(p_\pm - u) \ , \tag{36}$$

where we have subtracted the homogenous energy density of the underlying spin wave. From the explicit solutions (31), (32) we calculate the total energy and the quantities P, M:

$$\begin{aligned} E_\pm &= 2\sqrt{\gamma} - \frac{2h}{\sqrt{-\alpha}}\left(\tan^{-1}\left(\frac{\beta}{2\sqrt{-\alpha\gamma}}\right) \pm \frac{\pi}{2}\right) , && (37) \\ P_\pm &= \int dx\,(p_\pm\,(\partial_x q_\pm) - uk) \\ &= \left(\tan^{-1}\left(\frac{\frac{\beta}{2}\,(1+u) - \gamma}{\sqrt{\gamma}\,(v + k\,(1-u))}\right) \pm \frac{\pi}{2}\text{sign}\ (v + k\,(1-u))\right) \\ &\quad - \left(\tan^{-1}\left(\frac{\frac{\beta}{2}\,(1-u) + \gamma}{\sqrt{\gamma}\,(v - k\,(1+u))}\right) \pm \frac{\pi}{2}\text{sign}\ (v - k\,(1+u))\right) , && (38) \\ M_\pm &= \int dx\,(p_\pm - u) \\ &= \frac{2}{\sqrt{-\alpha}}\left(\tan^{-1}\left(\frac{\beta}{2\sqrt{-\alpha\gamma}}\right) \pm \frac{\pi}{2}\right) . && (39) \end{aligned}$$

In eqs. (38), (39) we have also subtracted the homogenous contributions of the nonlinear spin wave from the integrands in (5), (6). For vanishing magnetic field and a given velocity both solutions have the same energy. In particular, for $h = 0$, $\beta > 0$ the limiting $(+)$–soliton (34), (35) has the same energy as the $(-)$–soliton, i. e. the homogenous spin wave. Eq. (37), (38) provide a parametric representation of the dispersion law $E(P)$ in terms of the soliton velocity. In figure 3 we give an example for the function $E(P)$ for both soliton types.

3.2 The case $\alpha \geq 0$

For $\alpha > 0$ and $|u| < 1$ we have shown that the spin wave solutions are unstable. Moreover, as the integration of eq. (22) is formally the same as for $\alpha < 0$, we see from (31) that we need

$\gamma > 0$ for a localized solution. Because of (33) this cannot be achieved for nonnegative α. Thus no localized solution of the form (19) exists for $|u| < 1$

The case $|u| = 1$ is particular. The corresponding spin wave realizes the absolute minimum of $\mathcal{H}$ and the wavenumber k becomes irrelevant. For such boundary condition well–known solitons exist, which can be derived similarly by elementary integration (Kosevich *et al.* [12], Long and Bishop [13]). These solitons are characterized by two independent parameters v and Ω because the relation (28) is not valid if the boundary condition (23) is not imposed. Therefore the soliton can alternatively be described by two independent conserved quantities, the total momentum and the total angular momentum, and the exact dipersion law $E = E(P, M)$ can be obtained [12, 14]. These two–parametric solutions are also valid for $\alpha \leq 0$ and suffciently large $(h + \Omega)$, and under further conditions algebraic solitons arise as limiting cases [11].

4 Stability and scattering of solitons

We now want to establish the stabitlity of the solitary waves under interaction with magnons and under scattering with each other. The stability is clear if one can prove the integrability of the model for the generalized boundary conditions used here. The solitary waves have some remarkable similarity to dark solitons occuring as solutions to the Nonlinear Schrödinger equation with repulsive interaction [15]. The latter evolution equation is well–known to be integrable [16]. Moreover, Nakamura and Sasada [9] have proposed a gauge transformation mapping the Nonlinear Schrödinger equation with attractive (repulsive) interaction on the Heisenberg chain with easy–axis (easy–plane) anisotropy, where the boundary conditions to the spin models correspond to the absolute ground state. Unfortunately, for the easy–plane case which is of interest here, this work suffers from some errors pointed out by other authors [17, 18]. Thus, we shall use other means to examine the question of stability.

In order to provide a brief demonstration of the physical significance of the solitary waves presented before and also to study discreteness effects to these continuum solutions we have carried out numerical simulation for the discrete model (1). The spin dynamics is given by the following Landau–Lifshitz–equation:

$$\partial_t \vec{S}_n = \vec{S}_n \times \left(\vec{S}_{n-1} + \vec{S}_{n+1} \right) + h \left(\vec{S}_n \times \vec{e}_z \right) + \alpha \left(\vec{S}_n \cdot \vec{e}_z \right) \left(\vec{S}_n \times \vec{e}_z \right) \tag{40}$$

with $\vec{S}_n^2 = 1$ and $\vec{e}_z$ being the unit vector in z–direction. This equation may also be written in an explicitly Hamiltonian form, e. g. using the canonical conjugate variables $p_n = \cos \vartheta_n$, $q_n = \varphi_n$. It is seen easily that the discretized version of the nonlinear spin wave, i. e. $p_n = u$, $q_n = kn + \omega t + \text{const.}$, is an exact solution of the dicrete model [19]. The natural time unit of the system is chosen to be unity by the definition (1). We numerically integrated the equations of motion (40) on lattices of up to 5000 sites using a Runge–Kutta scheme of fourth order with a stepwidth of $0.01 \ldots 0.05$. The boundary conditions were implemented by rotating the spins on the terminal lattice sites with a constant frequency ω given by (8) while the z–component has the constant value u.

If one uses the one–soliton solution of the continuum model evaluated at the discrete lattice sites as initial data for a simulation, magnons are emitted from the soliton during the first few hundred time units. This is an discreteness effect since the continuum solitary wave can only be an approximate solution to the discrete model. After this process the soliton moves with constant velocity over the lattice, with no measurable sign of instability over several

thousand time units. The separation of magnons and solitons corresponds to the result in the continuum model that the admissible soliton velocities and magnon group velocities are strictly separated. In figure 4 we show an example where this effect is comparatively strong for a certain choice of parameters. Moreover, small magnons are also radiated from the edges of the system during the simulation.

Since the spin configuration outside a soliton decreases rapidly to the underlying nonlinear spin wave, different one–soliton solutions can be matched and scattering experiments can be carried out. If one uses appropriately matched continuum solitons as initial data, the two solitons can apparently pass each other keeping perfectly their shape and identity. More precisely, we found no effect of instability on the background of magnons that stem from the adjustment to the discrete system in the beginning of the simulation or are radiated from the edges. This holds for all values of the system parameters and energy of both (+)– and (−)–solitons. A typical example is shown in figure 5.

For more precise measurements the magnon background has to be reduced. This can be done by using the soliton shape after the adjustment to the lattice, i. e. the emission of magnons, as initial data for further simulations. Different spin configurations obtained by this procedure can be matched for a scattering experiment. This method leads to much purer initial conditions and works particularly well in the case $h = 0$, $k = 0$, where the spin configuration outside the solitons is spatially constant. In order to prevent the small magnons from the edges disturbing the scattering process one can choose a sufficiently large system. In figure 6 we show a typical scattering process. On the natural scale of the problem the two solitons apparently pass each other without any sign of instability. If the scale of the vertical axis is magnified by a factor of about 10^3 one clearly sees that the solitons do not perfectly keep their shape but magnons with amplitudes of order 10^{-4} are emitted as a result of the scattering process. We have repeated this experiment with different stepwidths of the Runge–Kutta scheme and confirmed that this effect is not a numerical artifact.

This observation strongly indicates the nonintegrability of the discrete model in the strict mathematical sense. Of course, this does not imply any statement about the integrability of the continuum model under the generalized boundary conditions. In particular not, since the discrete model shows properties very near to those of integrable models.

In summary, we have demonstrated by numerical simulations that our solitary waves are physically significant objects showing a highly distinct but not absolute stability in their time evolution.

5 Conclusions

In this work we have examined the classical Heisenberg chain with uniaxial anisotropy under generalized boundary conditions. For an anisotropy of easy–plane type two branches of solitons are found on the background of a nonlinear spin wave that is used as boundary condition. The parameter area of stability of the nonlinear spin wave solution coincides with the existence of the solitary waves. For the easy–axis and the isotropic model we have shown that no further one–soliton solutions exist apart from those already known. Moreover, in these cases the spin wave solutions are found to be unstable. This fact has apparently not been realized yet, since the spin wave solutions (7) have been discussed for such systems by various authors without mentioning their instability [1, 2, 4, 20].

For the case $\alpha < 0$ we have examined the stability of the continuum solitary waves by numerical simulations of the discrete Heisenberg chain. The solitary waves turn out to be highly but not absolutely stable in their time evolution and especially under scattering with each other. Thus the discrete model is shown to have properties very near to those typical for integrable models, but small effects indicating nonintegrability are clearly observed. From these numerical results and also from the similarity of the solitary waves obtained here to dark solitons of the Nonlinear Schrödinger equation one can strongly conjecture the integrability of the continuum model.
The spin wave state (7) could serve as a phenomenological model for a one–dimensional easy–plane magnet at finite (not necessarily very low) temperature. In thermal equilibrium the spatially averaged energy density and magnetization have certain values determining the parameters u, k. However, the long–range ordering in this one–dimensional system must be destroyed at finite temperature by thermal fluctuations [21]. The linear and localized nonlinear excitations obtained in this work provide a possible mechanism for this effect.

References

[1] H.-J. Mikeska, M. Steiner, Adv. Phys. **40**, 191 (1991)

[2] A. M. Kosevich, B. A. Ivanov, A. S. Kovalev, Phys. Rep. **194**, 117 (1990)

[3] L. A. Takhtajan, Phys. Lett. **64A**, 235 (1977)

[4] H. C. Fogedby, J. Phys. A: Math. Gen. **13**, 1467 (1980)

[5] E. K. Sklyanin, LOMI preprint (1979)

[6] V. E. Zakharov, L. A. Takhtadzhyan, Theor. Math. Phys. **38**, 17 (1979)

[7] M. Lakshmanan, Phys. Lett. **61A**, 53 (1977)

[8] G. R. W. Ouispel, H. W. Capel, Phys. Lett. **88A**, 371 (1982)

[9] K. Nakamura, T. Sasada, J. Phys. C: Solid State Phys. **15**, L915 (1982)

[10] I. A. Akhiezer, A. E. Borovik, Soviet Physics JETP **25**, 885 (1967)

[11] B. A. Ivanov, A. M. Kosevich, I. V. Manzhos, Solid State Physics, **34**, 417 (1980)

[12] A. M. Kosevich, B. A. Ivanov, A. S. Kovalev, JETP Lett. **25**, 486 (1977), Sov. J. Low Temp. Phys. **3**, 440 (1977)

[13] K. A. Long, A. R. Bishop, J. Phys. A: Math. Gen. **12**, 1325 (1979)

[14] T. Sasada, J. Phys. Soc. Jpn. **51**, 2446 (1982)

[15] See e. g. M. Remoissenet, *Waves Called Solitons*, 2nd Edition, Springer (1996)

[16] For a exhaustive treatment see L. D. Fadeev, L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer (1987)

[17] A. Kundu, O. Pashaev, J. Phys. C: Solid State Phys., **16**, L585 (1983)

[18] V. P. Kotlyarov, J. Phys C: Solid State Phys., **17**, L139 (1984)

[19] J. A. G. Roberts, C. J. Thompson, J. Phys. A: Math. Gen. **21**, 1769 (1988)

[20] M. Lakshmanan, T. W. Ruijgrok, C. J. Thompson, Physica **84a**, 577 (1976)

[21] N. D. Mermin, H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966), N. D. Mermin, J. Math. Phys. **8**, 1061 (1967)

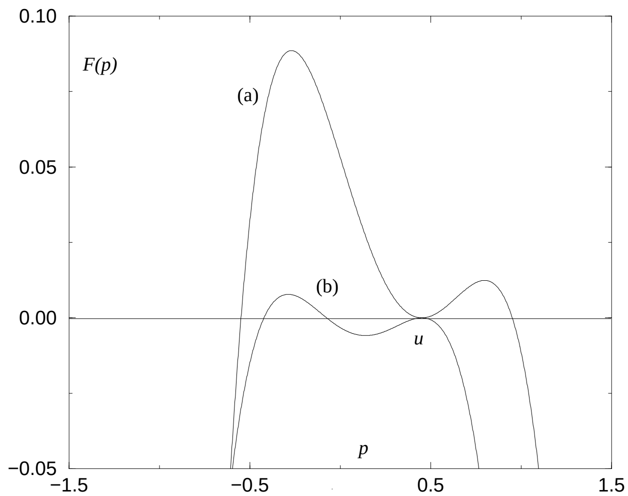

Figure 1: The typical behaviour of the polynomial $F(p)$.

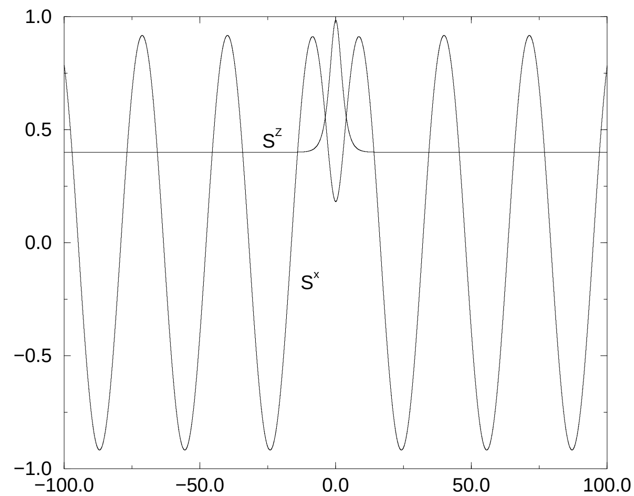

Figure 2: Spin field configuration of a (+)–soliton with system parameters $\alpha = -0.5$, $h = 0.1$, $u = 0.4$, $k = 0.2$ and soliton parameters $v = 0.4$, $x_0 = 0$, $q_0 = 0$ as a function of x at time $t = 0$.

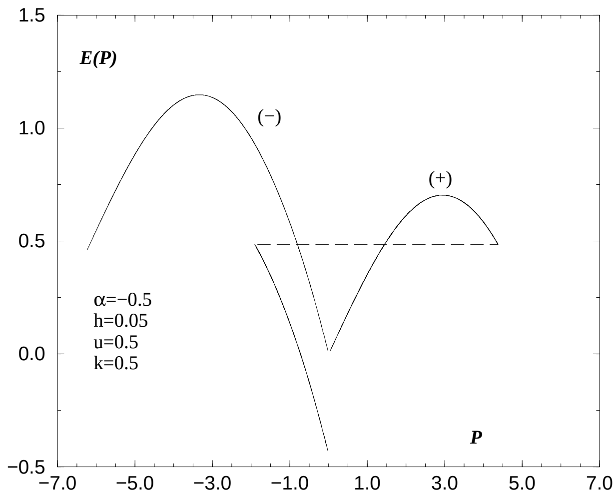

Figure 3: Dispersion law $E(P)$ for particular values of the system parameters. The branches of (+)– and (−)–solutions meet in the origin. The dashed line is a guide to the eye.

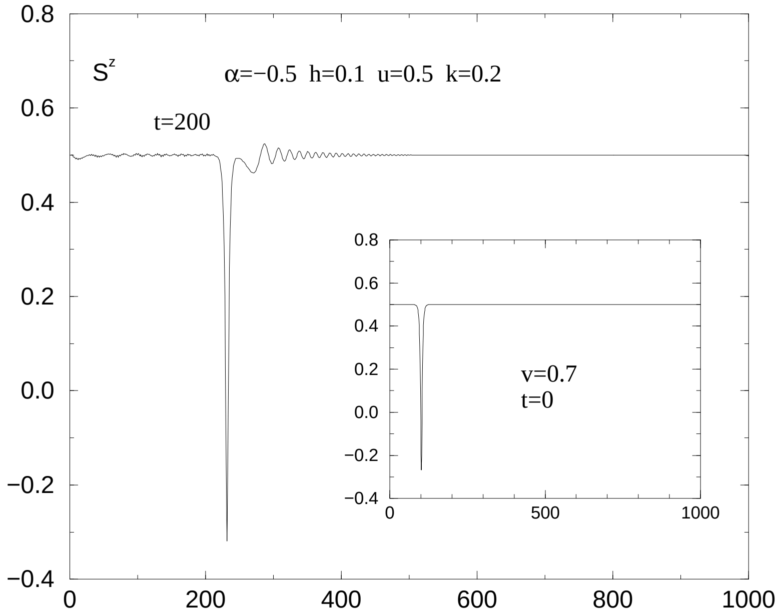

Figure 4: Time evolution with a continuum solitary wave as initial data (small graph, $(-)$–solution) on a lattice of 1000 sites. For this choice of parameters comparatively large magnons are emitted in the first few hundred time units; only the component S^z is shown.

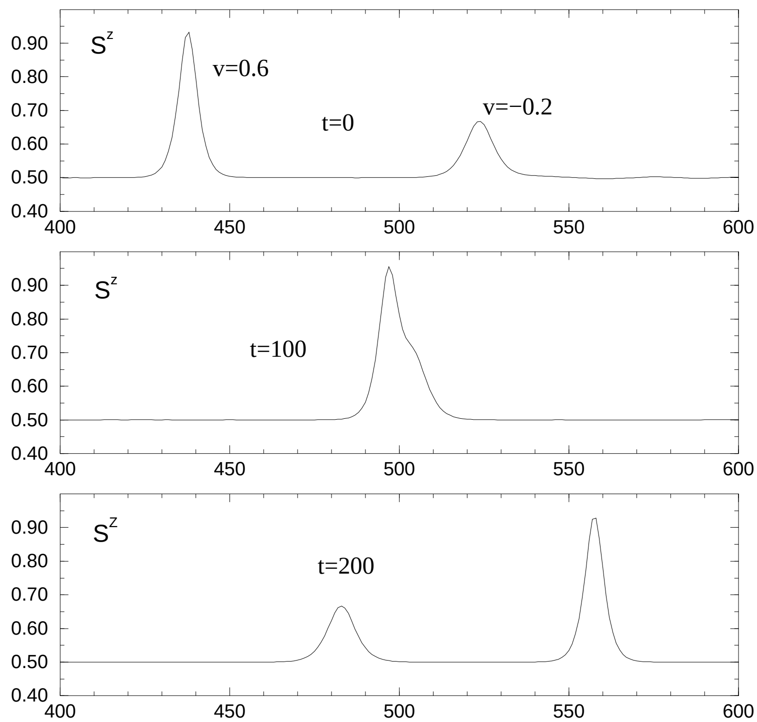

Figure 5: Soliton scattering in a system with $\alpha = -0.5$, $h = 0.2$, $u = 0.5$, $k = 0.25$; only the component S^z is shown.

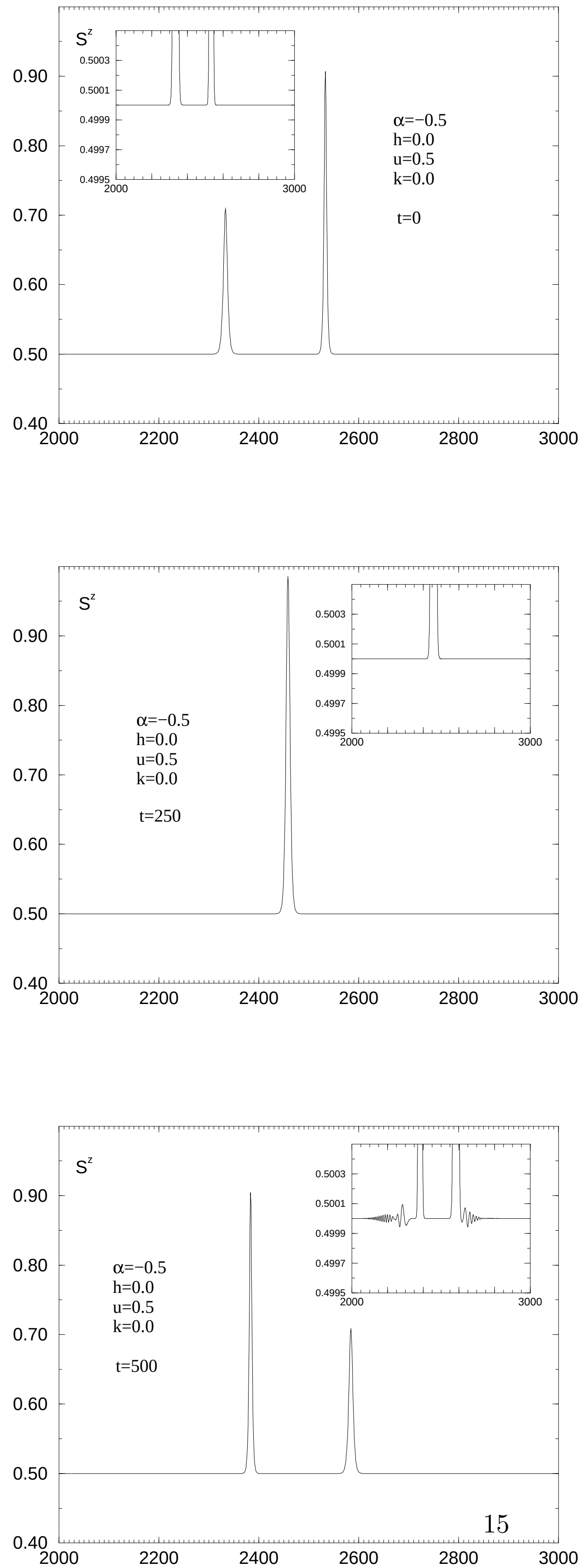

Figure 6: A scattering experiment with very pure initial conditions. In the insets the $z-$ component of the spins is plotted with a magnified scale showing the emission of very small